Math on a Sphere

Topic 3: Lines on the Sphere

In our previous exploration, we made the turtle draw three lines to end up with a closed shape. The figure below is the same one that we finished with in the previous exploration:

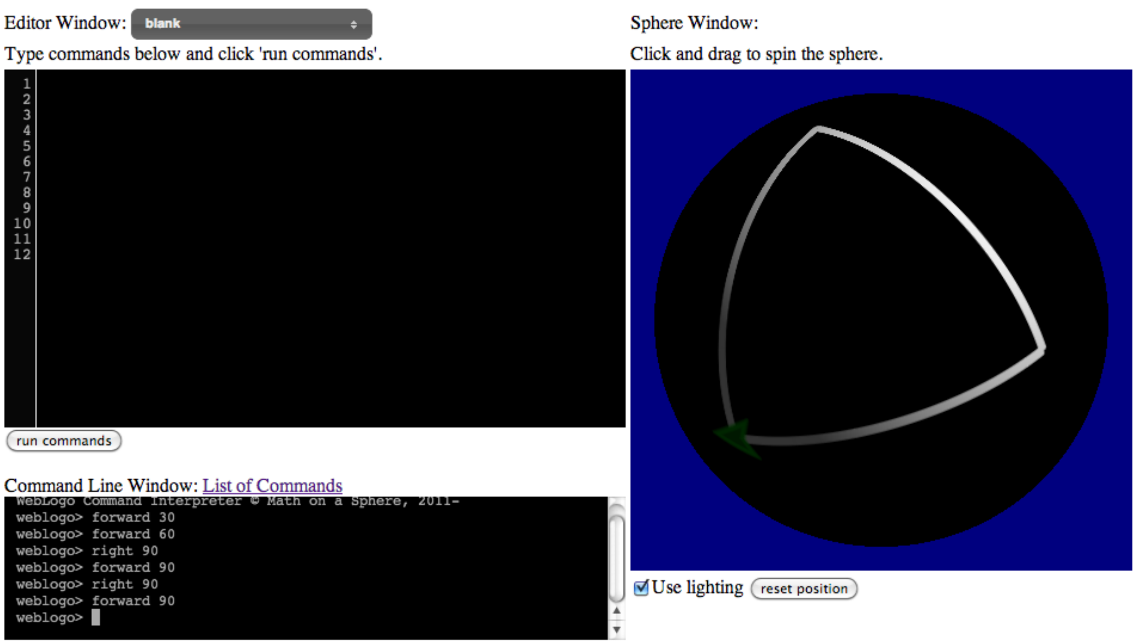

Figure 1. Three turtle lines.

Maybe all of this turtle-drawing has seemed natural so far, but that figure above is actually rather interesting, when you think about it. First of all, when we move the turtle "forward", what are we really doing? If you imagine the turtle as a little creature crawling about on the surface of the sphere, you could think of our turtle as taking a series of (very small) steps, one after another, in the direction that it's facing. Thus, when we start the turtle at the equator and move it 30 steps forward, we get a drawing like the one that we saw in the previous exploration (reproduced below):

Figure 2. The turtle has moved forward 30 steps from its original position on the equator.

Now, let's pause to ask what might seem like a strange question: we know that the turtle has just moved forward in the direction that it was headed... but has it made a *line*? Or, to put the matter another way, has it made a *straight* line? In one sense, viewing the sphere from the outside, we might think that the turtle *hasn't* made a straight line, since (again, viewing from the outside) the line drawn on the sphere looks curved to us: it bends from the equator upwards toward the North Pole. But in another sense–from the perspective of the turtle on the sphere–the line *could* be considered a "straight" line. After all, the turtle kept moving forward along the sphere, drawing the line as it moved; and from the turtle's perspective, it never once turned to the left or right as it was moving. So, as viewed by the turtle, our `forward 30` move has in fact produced a "straight line"!

This is a first opportunity to ponder what it really means for a line to be "straight". For a turtle, restricted to the surface of a sphere, moving in a straight line means moving forward without turning. From the perspective of a viewer standing outside the sphere and watching the turtle move, the turtle is drawing an *arc of a great circle*: that is, the turtle is moving along the sphere in such a way that if it keeps going, it'll just draw a circle that divides the sphere into two equal halves.

Let's try one more little experiment to see what this means. First, we'll use the `ca` command to clear the sphere drawing and return the turtle to its original position. Now, let's draw a "straight line" that moves the turtle in a full *great circle* around the sphere. In our Math on a Sphere system, 360 steps will move the turtle all the way around the sphere, so we type:

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forward 360
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Once we've typed this command, the turtle will have moved all the way around the sphere, and the screen will look as in the figure below:

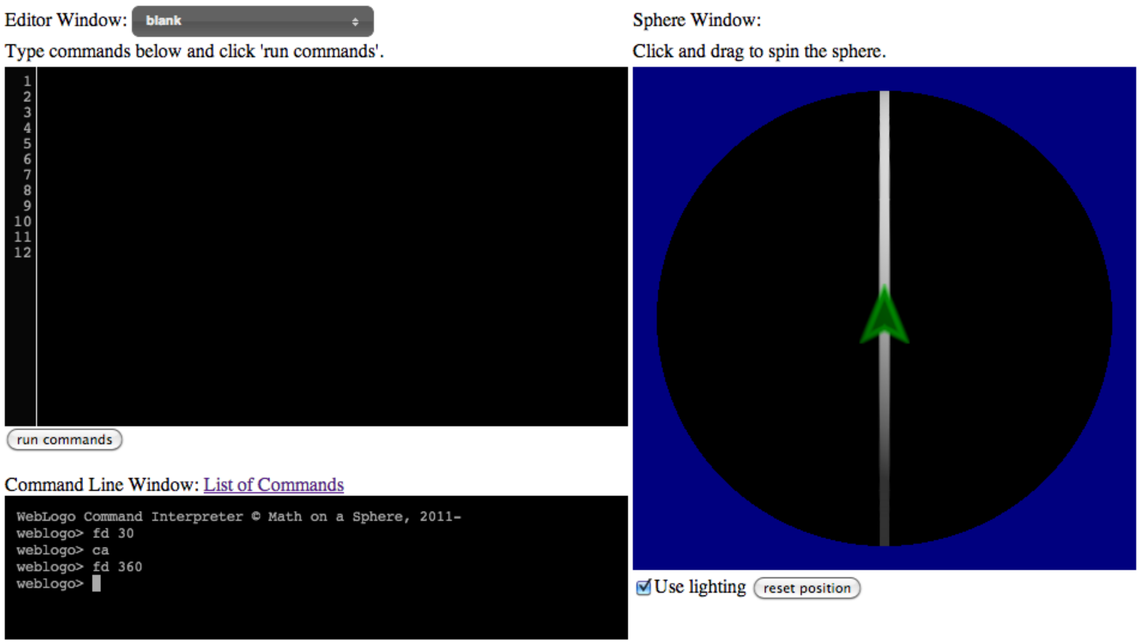

Figure 3. The turtle has moved in a full great circle around the sphere.

There's still another odd thing to ponder about this example. As far as the turtle is concerned, it just took a journey by moving forward 360 steps, without turning to the left or right; and after that entire journey, it is back in the very same position that it started! Our usual intuition is that if we move forward (say, on a flat surface), without turning, we'll always move further and further from our original starting position. On a sphere, however, if we move forward all the way around the sphere, we end up back in our starting position. In mathematics, the fact that the turtle can move all the way around the sphere and get back to its starting position is related to the fact that the sphere is *finite* (it doesn't go on forever, like an infinite plane), and it's a *closed* surface (it doesn't have any boundary).

Actually, our own planet Earth is (to a pretty good approximation) a huge sphere. If we were to play out the role of the turtle and start walking in a straight line, continually following our nose, we would–after about 25,000 miles of walking (and I guess swimming)–end up in the same spot that we began. Well, okay, we're not going to try that experiment–but when airplanes fly from one place to another, they do (again, to a good approximation) make the trip by flying in the arc of a great circle from one spot to another on the globe. We'll return to that topic later in these explorations.