Last updated on 2013-04-25

Math on a Sphere

Topic 7: Making Small Circles

In Topic 3, we learned how to make the turtle walk around a great circle on the sphere. Figure 1 summarizes this idea.

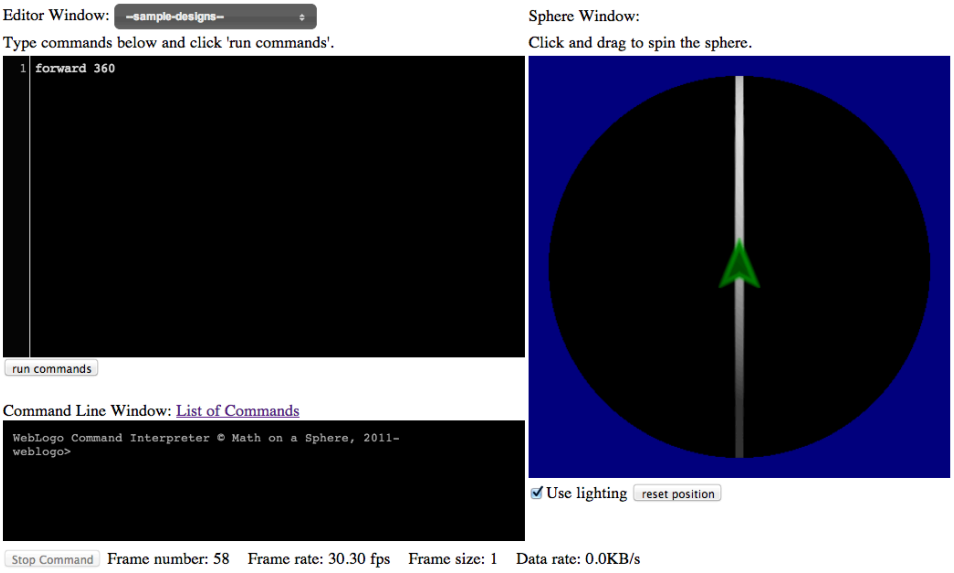


Figure 1. The turtle has moved in a great circle around the sphere. When viewed head-on, a great circle looks like a straight line.

(What's a great circle? If the turtle is walking along a great circle, the turtle is moving along the sphere in such a way that if it keeps going, it'll just draw a circle that divides the sphere into two equal halves. This idea was discussed in Topic 3.)

From the standard viewing angle, this great circle looks like a line. (Which makes sense, since a line around a sphere is a circle!) But if we rotate the sphere we can see the circular shape more easily, as shown in Figure 2.

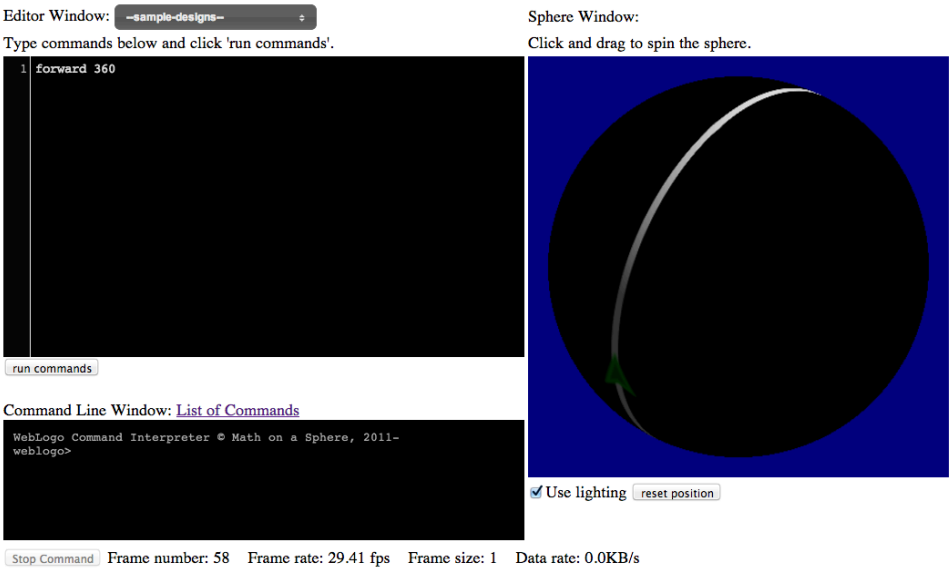


Figure 2. Rotating the sphere can make it easier to see what you've drawn. In this case, it helps us see that the great circle is really curved.

So, once we understand the basic Math on a Sphere commands, creating great circles isn't too difficult. But there are lots of circles that we can draw on a sphere that aren't great circles. How do we make these smaller circles?

Making small circles is pretty tricky at first. Let's go through the basic concepts involved so that we can understand why the turtle can't just walk forward to make small circles.

Let's start by thinking about what the turtle is doing when it is walking around a great circle. An example of a great circle that we're all pretty familiar with is the equator. So let's say that our turtle starts at the "south pole", walks down to the equator, and then traces out the equator. This example is shown below, in Figure 3.

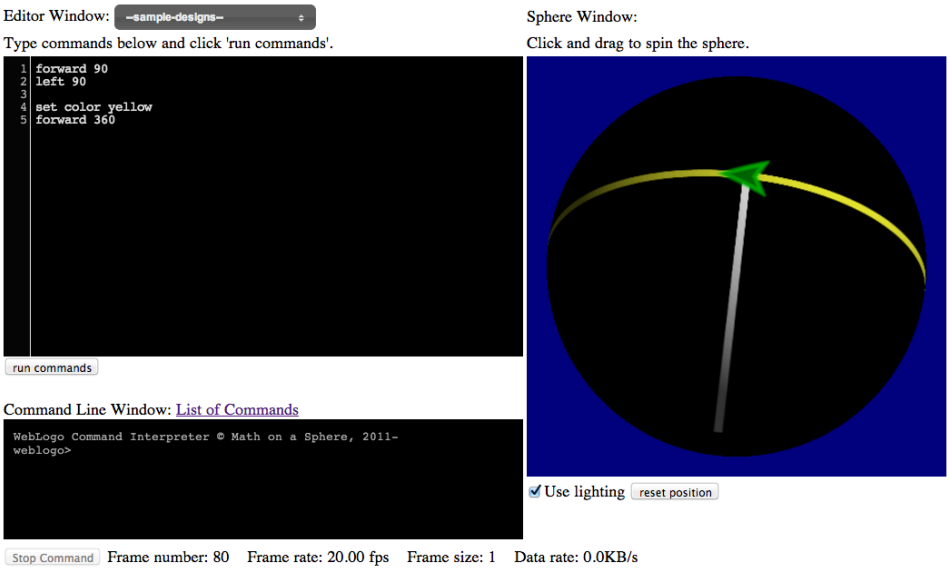


Figure 3. To get to the equator, the turtle can start from the "south pole", walk south 90 degrees, then turn left 90 degrees. The turtle can then trace out the equator (in yellow) by walking forward 360 degrees.

The instructions given to the turtle in Figure 3 are as follows:

```
forward 90
left 90

set color yellow
forward 360
```

As we mentioned earlier, the turtle can't make small circles in the same way that it makes great circles. That is, the turtle can't just walk forward. To understand why, let's try to trace out a latitude line. This is a good test because latitude lines are not great circles. (Want to know more about latitude lines? Consult Topic #6.)

Figure 4 shows what happens when we try to make a small circle by just walking forward. Figure 4 uses the same code as Figure 3 to make the equator, then uses additional lines of code to draw the pink circle. Specifically, the turtle draws out the equator, returns to the "south pole", then goes to a latitude position of 45 degrees. The turtle turns 90 degrees, and begins to walk forward. If this method did produce the 45 degree latitude circle, the turtle would create a small circle parallel to the equator. But this is not what happens. Instead, the turtle produces another great circle that is at an angle to the equator.

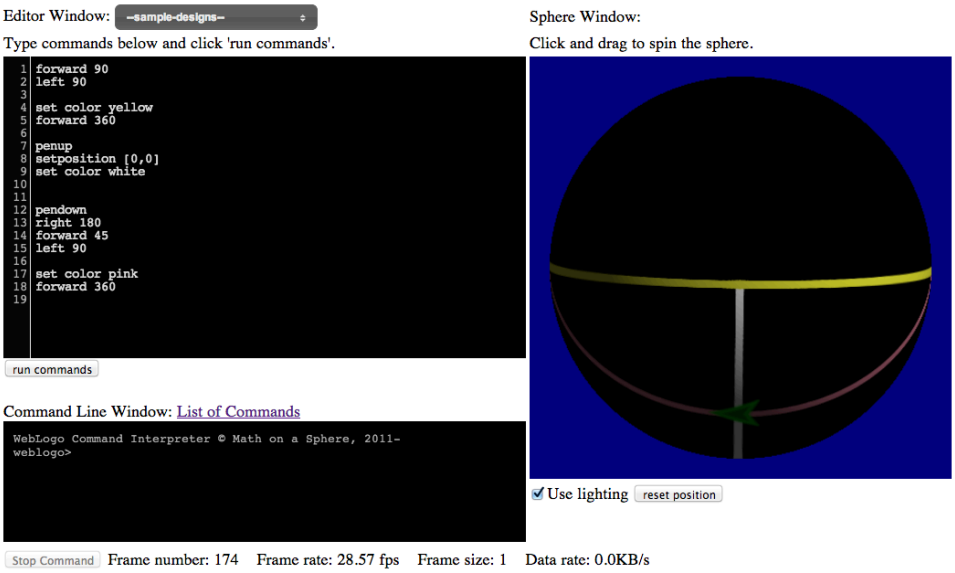


Figure 4. You can't make a small circle by just walking the turtle forward. The yellow circle is the equator (which is a great circle). The pink circle was created when we tried to make a small circle. *Code is provided at the end of this document.*

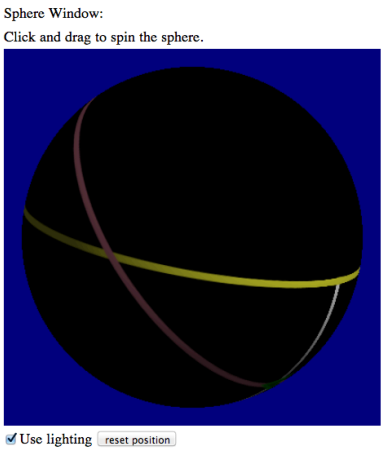
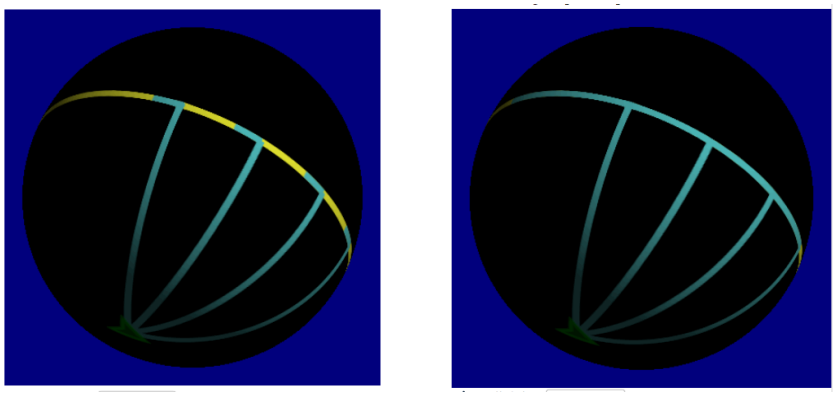


Figure 5. The same design as Figure 4, but with the sphere rotated.

So why didn't this work? We can get an idea of what's going on by considering the idea of tangents. A *tangent* to a circle is a line in the plane of the circle that touches the circle at a single point. A useful property of a tangent is that it is perpendicular to the radius drawn to the point of tangency. (The *point of tangency* is the point where the tangent and circle intersect. When two things are perpendicular, the angle between them is 90 degrees.)

We bring up the idea of tangents because it helps us understand what the turtle is doing as it draws small circles on the sphere. Each time the turtle takes a step forward, it's walking along a very small tangent to the circle. So, when we draw a circle on the sphere, we're basically tracing a bunch of these tangents. Therefore, to draw a circle we first have to think about what the tangents look like.

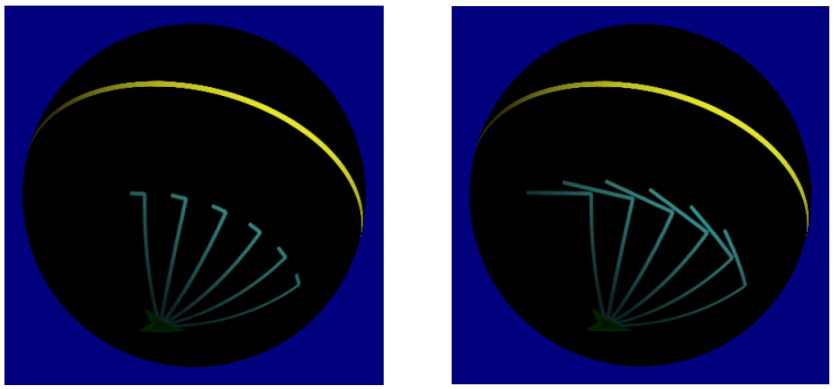
In the case of great circles, like the equator, figuring out the tangents is easy. If we draw a bunch of longitude lines, which are all great circles, we'd see that they all form a 90 degree angle with the equator. This is shown in Figure 6a. Now, if we make these tangent lines a little longer, we see that they all overlap (Figure 6b). This is why it is so easy to make great circle. Once we get on the first tangent, we can keep moving the turtle forward and it'll just walk onto the next tangent, and so on.



A B

Figure 6. Figuring out tangent lines for the equator. *Code is provided at the end of this document.*

Now, we can compare this to a different latitude line (the 40 degree latitude line).



A B

Figure 7. Figuring out tangent lines for a small circle (the 40 degree latitude line). *Code is provided at the end of this document.*

As you can see in Figure 7, when the lines are short they seem like they could overlap and form a circle (Figure 7A). But when we extend them we see that they really don't overlap (Figure 7B). So if the turtle kept walking along one of these lines, the turtle would become very off course from the original circle we intended for the turtle to walk around. Thus, if we want the turtle to stay on track we must move it forward a little, turn a little, move forward a little, turn a little, and so on. (In other words, since the heading of the tangent line is changes continuously around the circle, the turtle needs to keep turning.) This concept is illustrated in Figure 8.

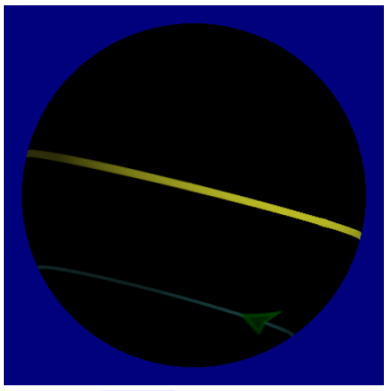


Figure 8. We can make small circles by having the turtle walk for a little, turn, walk, turn, and so on.

Now play around and make your own small circles! What happens if you change the amount the turtle turns, or walks forward? What if you change the number of times the turtle repeats the instructions?

Code for Figure 4:

```
forward 90
left 90

set color yellow      // draws the equator
forward 360

penup
setposition [0,0]     // returns to the starting position
set color white

pendown
right 180
forward 45
left 90

set color pink        // tries to draw a small circle
forward 360
```

Code for Figure 6a:

```
set color yellow
penup
forward 90
left 90

pendown
forward 360

penup
left 90
forward 90
right 180
pendown

set color cyan

repeat 4 {
      forward 90
         left 90
         forward 10
         penup
         setposition [0,0]
         right 180
         pendown
         right 40
         }
```

Code for Figure 6b:

```
set color yellow
penup
forward 90
left 90

pendown
forward 360

penup
left 90
forward 90
right 180
pendown

set color cyan

repeat 4 {
      forward 90
         left 90
         forward 50
```

```
        penup
        setposition [0,0]
        right 180
        pendown
        right 80
        }
```

Code for Figure 7a

```
set color yellow
penup
forward 90
left 90

pendown
forward 360

penup
left 90
forward 90
right 180
pendown

set color cyan
set pensize 1

repeat 6 {
        forward 50
         left 90
         forward 5
         penup
         setposition [0,0]
         right 180
         pendown
         right 25
         }
```

Code for Figure 7b

```
set color yellow
penup
forward 90
left 90

pendown
forward 360

penup
left 90
forward 90
right 180
pendown

set color cyan
set pensize 1

repeat 6 {
        forward 50
         left 90
         forward 25
         penup
         setposition [0,0]
         right 180
         pendown
         right 50
         }
```

Code for Figure 8

```
set color yellow
penup
forward 90
left 90

pendown
forward 360

penup
left 90
forward 90
right 180
pendown

set color cyan
set pensize 1
penup
forward 50

pendown
left 90

repeat 56 {
      forward 5
         left 4.2
         }
```